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# Wolfe type second-order symmetric duality in nondifferentiable programming

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**Abstract**

A pair of Wolfe type second-order symmetric dual programs involving nondifferentiable functions is considered and appropriate duality theorems are established under $\eta_1$-bonvexity/$\eta_2$-boncavity. Several known results including that of Mond and Gulati et al. are obtained as special cases.
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*Keywords:* Symmetric duality; Wolfe type second-order duality; $\eta$-Bonvexity; Support function; Nondifferentiable programming

## 1. Introduction

Dantzig et al. [4], Mond [8] and Bazaraa and Goode [2] studied symmetric duality in nonlinear programming assuming the kernel function $f(x, y)$ to be convex in $x$ and concave in $y$. Subsequently, Mond and Weir [11] presented a distinct pair of symmetric dual nonlinear programs which admits the relaxation of the convexity/concavity assumption to pseudoconvexity/pseudoconcavity.

Mangasarian [7] introduced the concept of second-order duality. Mond [9] established Mangasarian's duality relations assuming $f(x, y)$ to be bonvex/boncave [3]. Mond and

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Weir [12] stating computational usefulness, have presented a different second-order dual. Mond [9] has also discussed second-order symmetric dual programs. Gulati et al. [5] studied two distinct pairs of second-order symmetric dual problems under generalized bonvexity/boncavity assumptions.

The work cited in above papers involved differentiable functions. Second-order symmetric duality involving nondifferentiable functions has been discussed by Hou and Yang [6] for Mond–Weir type dual, and by Ahmad and Husain [1] and Yang et al. [15] for Wolfe type dual.

In the present paper, we consider a pair of Wolfe type second-order symmetric dual programs involving nondifferentiable functions and prove duality relations using $\eta_1$-bonvexity/$\eta_2$-boncavity. Our results relax the nonnegativity conditions in the problems studied by Yang et al. [15] and subsume the work in [1,4,5,7,9].

## 2. Notations and preliminaries

Let $R^n$ denote the $n$-dimensional Euclidean space. Let $f(x, y)$ be a real valued twice differentiable function defined on an open set in $R^n \times R^m$, and $\nabla_x f(\bar{x}, \bar{y})$ denote the gradient vector of $f$ with respect to $x$ at $(\bar{x}, \bar{y})$. $\nabla_y f(\bar{x}, \bar{y})$ is defined similarly. Also, let $\nabla_{xx} f(\bar{x}, \bar{y})$ and $\nabla_{yy} f(\bar{x}, \bar{y})$ denote the $n \times n$ and $m \times m$ symmetric Hessian matrices at $(\bar{x}, \bar{y})$, respectively.

**Definition 1.** A real twice differentiable function $f$ defined on a set $X \times Y$, where $X$ and $Y$ are open sets in $R^n$ and $R^m$, respectively, is said to be $\eta_1$-bonvex in the first variable at $u \in X$, if there exists a function $\eta_1 : X \times X \to R^n$ such that for $v \in Y$, $r \in R^n$, $x \in X$,

$$f(x, v) - f(u, v) \geqslant {\eta_1}^T(x, u)\big[\nabla_x f(u, v) + \nabla_{xx} f(u, v)r\big] - \frac{1}{2} r^T \nabla_{xx} f(u, v)r,$$

and $f(x, y)$ is said to be $\eta_2$-bonvex in the second variable at $v \in Y$, if there exists a function $\eta_2 : Y \times Y \to R^m$ such that for $u \in X$, $p \in R^m$, $y \in Y$,

$$f(u, y) - f(u, v) \geqslant {\eta_2}^T(y, v)\big[\nabla_y f(u, v) + \nabla_{yy} f(u, v)p\big] - \frac{1}{2} p^T \nabla_{yy} f(u, v)p.$$

A twice differentiable function $f$ is $\eta$-boncave if $-f$ is $\eta$-bonvex.

It has been revealed in [13] by means of an example that the above class of functions is an extension of the bonvex functions. For $r$ and $p$ to be zero vectors, the above inequalities were introduced by Mond and Hanson [10].

**Definition 2.** Let $C$ be a compact convex set in $R^n$. The support function of $C$ is defined by

$$S(x \mid C) = \max\{x^T y\colon\ y \in C\}.$$

A support function, being convex and everywhere finite, has a subdifferential, that is, there exists $z \in R^n$ such that

$$S(y \mid C) \geqslant S(x \mid C) + z^T(y - x) \quad \text{for all } y \in C.$$

The subdifferential of $S(x \mid C)$ is given by

$$\partial S(x \mid C) = \{z \in C\colon\ z^T x = S(x \mid C)\}.$$

For any set $S \subset R^n$ the normal cone to $S$ at a point $x \in S$ is defined by

$$N_S(x) = \{y \in R^n\colon\ y^T(z - x) \leqslant 0 \text{ for all } z \in S\}.$$

It can be easily seen that for a compact convex set $C$, $y$ is in $N_C(x)$ if and only if $S(y \mid C) = x^T y$, or equivalently, $x$ is in $\partial S(y \mid C)$.

We now consider the following Wolfe type symmetric dual program:

**Primal problem (PP):**

$$\begin{aligned}
\text{minimize}\quad & F(x,y,p) = f(x,y) + S(x \mid C) - \frac{1}{2}p^T \nabla_{yy} f(x,y)p \\
& \qquad - y^T \nabla_y f(x,y) - y^T \nabla_{yy} f(x,y)p \\
\text{subject to}\quad & \nabla_y f(x,y) - z + \nabla_{yy} f(x,y)p \leqslant 0, && (1)\\
& z \in D. && (2)
\end{aligned}$$

**Dual problem (DP):**

$$\begin{aligned}
\text{maximize}\quad & G(u,v,r) = f(u,v) - S(v|D) - \frac{1}{2}r^T \nabla_{xx} f(u,v)r \\
& \qquad - u^T \nabla_x f(u,v) - u^T \nabla_{xx} f(u,v)r \\
\text{subject to}\quad & \nabla_x f(u,v) + w + \nabla_{xx} f(u,v)r \geqslant 0, && (3)\\
& w \in C, && (4)
\end{aligned}$$

where

(1) $f$ is a differentiable function from $R^n \times R^m \to R$,
(2) $r, w$ are vectors in $R^n$ and $p, z$ are vectors in $R^m$, and
(3) $C$ and $D$ are compact convex sets in $R^n$ and $R^m$, respectively.

## 3. Symmetric duality

We prove the following duality results for the pair of problems (PP) and (DP).

**Theorem 3.1** (Weak duality). *Let $(x, y, z, p)$ be feasible for the primal problem* (PP) *and $(u, v, w, r)$ be feasible for the dual problem* (DP). *Let*

(i) *$f(.,v) + (.)^T w$ be $\eta_1$-bonvex in the first variable at $u$,*
(ii) *$f(x,.) - (.)^T z$ be $\eta_2$-boncave in the second variable at $y$,*
(iii) *$\eta_1(x,u) + u \geqslant 0$ and $\eta_2(v,y) + y \geqslant 0$.*

*Then*

$$F(x, y, p) \geqslant G(u, v, r).$$

**Proof.** By the hypotheses (i) and (ii),

$$\begin{aligned} & f(x, v) + x^T w - f(u, v) - u^T w \\ & \quad \geqslant \eta_1^T(x, u)\big[\nabla_x f(u, v) + w + \nabla_{xx} f(u, v) r\big] - \frac{1}{2} r^T \nabla_{xx} f(u, v) r \end{aligned}$$

and

$$\begin{aligned} & f(x, y) - y^T z - f(x, v) + v^T z \\ & \quad \geqslant -\eta_2^T(v, y)\big[\nabla_y f(x, y) - z + \nabla_{yy} f(x, y) p\big] + \frac{1}{2} p^T \nabla_{yy} f(x, y) p. \end{aligned}$$

Adding these inequalities, we get

$$\begin{aligned} & f(x, y) - f(u, v) - \frac{1}{2} p^T \nabla_{yy} f(x, y) p \\ & \qquad + \frac{1}{2} r^T \nabla_{xx} f(u, v) r + x^T w - u^T w - y^T z + v^T z \\ & \quad \geqslant \eta_1^T(x, u)\big[\nabla_x f(u, v) + w + \nabla_{xx} f(u, v) r\big] \\ & \qquad - \eta_2^T(v, y)\big[\nabla_y f(x, y) - z + \nabla_{yy} f(x, y) p\big], \end{aligned}$$

or

$$\begin{aligned} & f(x, y) - y^T\big[\nabla_y f(x, y) - z + \nabla_{yy} f(x, y) p\big] - \frac{1}{2} p^T \nabla_{yy} f(x, y) p \\ & \qquad + \frac{1}{2} r^T \nabla_{xx} f(u, v) r - f(u, v) + u^T\big[\nabla_x f(u, v) + w + \nabla_{xx} f(u, v) r\big] + x^T w \\ & \qquad - u^T w - y^T z + v^T z \\ & \quad \geqslant \big(\eta_1(x, u) + u\big)^T\big[\nabla_x f(u, v) + w + \nabla_{xx} f(u, v) r\big] \\ & \qquad - \big(\eta_2(v, y) + y\big)^T\big[\nabla_y f(x, y) - z + \nabla_{yy} f(x, y) p\big] \\ & \quad \geqslant 0 \quad \big(\text{using (1), (3) and hypothesis (iii)}\big). \end{aligned}$$

Finally, since $x^T w \leqslant S(x \mid C)$ and $v^T z \leqslant S(v \mid D)$, the last inequality yields

$$\begin{aligned} & f(x, y) + S(x \mid C) - \frac{1}{2} p^T \nabla_{yy} f(x, y) p - y^T \nabla_y f(x, y) - y^T \nabla_{yy} f(x, y) p \\ & \quad \geqslant f(u, v) - S(v \mid D) - \frac{1}{2} r^T \nabla_{xx} f(u, v) r - u^T \nabla_x f(u, v) - u^T \nabla_{xx} f(u, v) r, \end{aligned}$$

or $F(x, y, p) \geqslant G(u, v, r)$. □

**Theorem 3.2** (Strong duality). *Let $f : R^n \times R^m \to R$ be thrice differentiable and let $(\bar{x}, \bar{y}, \bar{z}, \bar{p})$ be a local optimal solution for* (PP). *If*

(i) $\nabla_{yy} f(\bar{x}, \bar{y})$ *is nonsingular, and*
(ii) *one of the matrices* $\frac{\partial}{\partial y_i}(\nabla_{yy} f)$, $i = 1, 2, \ldots, m$, *is positive or negative definite,*

*then* $\bar{p} = 0$, *there exists* $\bar{w} \in C$ *such that* $(\bar{x}, \bar{y}, \bar{w}, \bar{r} = 0)$ *is feasible for* (DP) *and*

$$F(\bar{x}, \bar{y}, \bar{p}) = G(\bar{x}, \bar{y}, \bar{r}).$$

*Also, if the hypotheses of Theorem* 3.1 *are satisfied for all feasible solutions of* (PP) *and* (DP), *then* $(\bar{x}, \bar{y}, \bar{z}, \bar{p})$ *and* $(\bar{x}, \bar{y}, \bar{w}, \bar{r})$ *are global optimal solutions for* (PP) *and* (DP), *respectively.*

**Proof.** Since $(\bar{x}, \bar{y}, \bar{z}, \bar{p})$ is a local optimal solution of (PP), there exist $\alpha \in R$, $\beta \in R^m$, $\gamma \in R^n$ such that following Fritz John conditions [14] are satisfied at $(\bar{x}, \bar{y}, \bar{z}, \bar{p})$ (for simplicity, we write $\nabla_x f, \nabla_{xy} f$ instead of $\nabla_x f(\bar{x}, \bar{y}), \nabla_{xy} f(\bar{x}, \bar{y})$, etc.):

$$\alpha(\nabla_x f + \gamma) + (\nabla_{xy} f)(\beta - \alpha\bar{y}) + \nabla_x[\nabla_{yy} f\,\bar{p}]\left(\beta - \alpha\bar{y} - \frac{1}{2}\alpha\bar{p}\right) = 0, \tag{5}$$

$$(\nabla_{yy} f)(-\alpha\bar{y} - \alpha\bar{p} + \beta) + \nabla_y[\nabla_{yy} f\,\bar{p}]\left(\beta - \frac{1}{2}\alpha\bar{p} - \alpha\bar{y}\right) = 0, \tag{6}$$

$$(\nabla_{yy} f)(\alpha\bar{p} - \beta + \alpha\bar{y}) = 0, \tag{7}$$

$$\beta^T(\nabla_y f - \bar{z} + \nabla_{yy} f\,\bar{p}) = 0, \tag{8}$$

$$\beta \in N_D(\bar{z}), \tag{9}$$

$$\gamma \in C, \quad \gamma^T\bar{x} = S(\bar{x}/C), \tag{10}$$

$$(\alpha, \beta) \neq 0, \tag{11}$$

$$(\alpha, \beta) \geqslant 0. \tag{12}$$

By hypothesis (i), (7) gives

$$\beta = \alpha(\bar{p} + \bar{y}). \tag{13}$$

Suppose $\alpha = 0$, then (13) implies

$$\beta = 0,$$

which contradicts (11). Hence

$$\alpha > 0. \tag{14}$$

Therefore, from (6),

$$(\nabla_y)[\nabla_{yy} f\,\bar{p}]\left(\frac{1}{2}\alpha\bar{p}\right) = 0,$$

which by hypothesis (ii) and (14) yields

$$\bar{p} = 0. \tag{15}$$

Now, (13) gives

$$\beta = \alpha\bar{y}. \tag{16}$$

Also, from (5) and (15)

$$\nabla_x f + \gamma = 0, \tag{17}$$

and from (8) and (15)

$$\beta^T(\nabla_y f - \bar{z}) = 0.$$

Using (14) and (16), the last equation yields

$$\bar{y}^T \nabla_y f = \bar{y}^T \bar{z}. \tag{18}$$

Now, taking $\bar{w} = \gamma \in C$ in (17), we find that $(\bar{x}, \bar{y}, \bar{w}, \bar{r} = 0)$ satisfies the constraints (3) and (4), of (DP), and is therefore a feasible solution for the dual problem (DP).

Moreover, since $\beta = \alpha \bar{y}$ and $\alpha > 0$, (9) implies $\bar{y} \in N_D(\bar{z})$, so that

$$\bar{y}^T \bar{z} = S(\bar{y} \mid D). \tag{19}$$

Therefore, using (10), (17)–(19), we get

$$\begin{aligned} &f(\bar{x}, \bar{y}) + S(\bar{x} \mid C) - \frac{1}{2}\bar{p}^T \nabla_{yy} f(\bar{x}, \bar{y})\bar{p} - \bar{y}^T \nabla_y f(\bar{x}, \bar{y}) - \bar{y}^T \nabla_{yy} f(\bar{x}, \bar{y})\bar{p} \\ &\quad = f(\bar{x}, \bar{y}) - S(\bar{y} \mid D) - \frac{1}{2}\bar{r}^T \nabla_{xx} f(\bar{x}, \bar{y})\bar{r} - \bar{x}^T \nabla_x f(\bar{x}, \bar{y}) - \bar{x}^T \nabla_{xx} f(\bar{x}, \bar{y})\bar{r}, \end{aligned}$$

that is,

$$F(\bar{x}, \bar{y}, \bar{p}) = G(\bar{x}, \bar{y}, \bar{r}).$$

Finally, from Theorem 3.1, we get that $(\bar{x}, \bar{y}, \bar{z}, \bar{p})$ and $(\bar{x}, \bar{y}, \bar{w}, \bar{r})$ are global optimal solutions for (PP) and (DP), respectively. □

## 4. Special cases

In this section, we consider some special cases of the problem (PP) and (DP) by choosing particular forms of the $\eta_1$ and $\eta_2$ and compact convex sets $C$ and $D$.

1. If $C = \{0\}$ and $D = \{0\}$, $\eta_1(x, u) = x - u$, $\eta_2(v, y) = v - y$, then the hypothesis (iii) becomes $x \geqslant 0$, $v \geqslant 0$, and so our problems (PP) and (DP) reduce to programs studied in Gulati et al. [5].
2. If we take $C = \{Ay\colon\ y^T Ay \leqslant 1\}$, $D = \{Bx\colon\ x^T Bx \leqslant 1\}$ and $\eta_1(x, u) = x - u$, $\eta_2(v, y) = v - y$, where $A$ and $B$ are positive semidefinite matrices, then $(x^T Ax)^{1/2} = S(x \mid C)$ and $(y^T By)^{1/2} = S(y \mid D)$. In this case (PP) and (DP) reduce to the problems considered in Ahmad and Husain [1].
3. As in Gulati et al. [5], the symmetric dual problems of [4] and the second-order dual problems studied by Mangasarian [7] and Mond [9] can also be obtained as special cases.

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